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The report on doctoral dissertation *Discrete neuron models from the point of view of dynamical system theory* by *Frank Fernando Llovera Trujillo*

I think that the submitted doctoral dissertation **fulfils the requirements for PhD degree**.

The dissertation consists of four published articles, three of them have been co-authored by both supervisors of the applicant.

In the papers entering in the dissertation authors apply advanced results from the dynamics of maps on the interval to 1D discrete models of neuron dynamics. This appears to be the **main scientific novelty**. It is however not clear to me what is the use of this analysis in the context of understanding mathematically the neurons behavior itself. In the thesis three discrete neuron models are considered: the Chalvo model, the CNV model -piecewise linear and its cubic version (in fact 1D reductions of these two-dimensional models). It is not clear to me which one is better? What kind of behavior the model should reproduce and explain? Why we have so many models? Does this mean that none of them is good enough?

Another (missing) mathematical aspect of the submitted dissertation (discussed on rather heuristic level, only) is: what kind of mathematical results survive when we pass of from 1D model to 2D-model? It would interesting to see also some results in this direction.

1 Brief summary of results

There are two fundamental types of 1D maps appearing in the dissertation

- unimodal and multimodal maps, with negative Schwartzian derivative
- maps called Lorenz maps , maps of interval with singularity modeled on 1D models of the Lorenz system

To these maps various tools from the modern theory of 1D maps has been applied to investigate the following phenomena

- saddle-node and period doubling bifurcations
- existence of attractors and their nature
- existence of chaos, in the topological sense (in the sense of Devaney, Li and Yorke, Block and Coppel) or the presence stochastic acip

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- characterisation of periodic orbits in terms pattern they generate using the rotation numbers

In the dissertation authors sometimes state some new theorems, but in all cases these are straightforward applications of some general and deep results from 1D dynamics and have few-line proofs.

2 Some minor comments

Below I include some more specific comments regarding some aspects of the dissertation:

- In paper **A:** devoted to 1D-Chialvo model. In discussing the bifurcations authors investigate a two parameter family depending of (r, k). They discuss the bifurcation of fixed points with fixed k treating r as varying parameter and things are nice, while when r is fixed then only the fold bifurcation is treated and regarding the period-doubling (the flip bifurcation) it is said that *fundamental condition for the occurrence of a flip bifurcation is not satisfied here, however the analysis the analysis of a fold bifurcation ... is still possible.* I find this statement rather unsatisfactory, while it is true that $\frac{\partial f}{\partial x \partial k} = 0$, the fixed point exists with multiplier equal to -1 and it might be interesting to see what happen in the neighborhood in two parameter family. Observe that the flip bifurcation happens as r is changing. The analysis of bifurcations is restricted to the bifurcations of fixed points in view of analytical difficulties in handling of periodic orbits. Apparently no attempt is made to explain bifurcations of periodic points seen in Fig. 7 and I've an impression that the bifurcations of fixed points analysed in the paper are not even in this figure. Are these fixed point bifurcation considered in the paper important for this model in explanation of some phenomena? Which phenomena?
- Regarding paper **D**: It is devoted computing periodic orbits itineraries for Lorenz-like maps. I would like to point out that a similar in sprit analysis has been done with a successful implementation (i.e. providing computer assisted proofs) for the full Lorenz system in paper by Z. Galias, W Tucker, *Validated study of the existence of short cycles for chaotic systems using symbolic dynamics and interval tools*, International Journal of Bifurcation and Chaos 21 (02), 551-563, 2011. In that paper authors first do a symbolic analysis of 1D-map model of the Lorenz attractor and then using this information the authors have found all periodic orbits up to period 16 (for some standard Poincaré map).
- the references in the summary to theorems in papers quite often do not agree (for example Theorem 3.7 in summary points to Theorem 4.6 in A, which should be Thm. 10 in A). Also some notation appearing in the summary are not defined (for example f_1, f_2, f_3 in Theorems 2.7, 2.12), but of course they are defined in included papers.

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